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CORRECTLY SOLVABLE PROBLEMS FOR THE LAPLACE-BELTRAMI
OPERATOR ON A RIEMANNIAN SPHERE WITH CUTS

ABSTRACT

of the PhD thesis for the degree of doctor of Philosophy (PhD)
in the specialty «8D05401-Математика»

Actuality of the research theme. The first successes of the theory of general boundary value problems for elliptic equations are due to the method of potentials, which made it possible to equivalently reduce the boundary value problem to a regular integral equation. In particular, the application of Fredholm's theory to boundary value problems made it possible to assert that the kernel and cokernel of the operator of the boundary value problem are finite-dimensional.

Taking into account the above sources, set the **main goal** of this thesis.

To achieve our goals, set the following tasks:

1. Consider the Laplace-Beltrami operator on a two-dimensional sphere from which one point has been removed. Construct the Green's function and write a correctly solvable problem. Explore the spectrum of the operator.
2. Consider the Laplace-Beltrami operator on a two-dimensional sphere from which an arc is cut out. Based on the results already obtained, construct a correctly solvable problem.
3. Consider the Laplace-Beltrami operator on a three-dimensional sphere from a four-dimensional space, from which a circle is cut out, which is not a hyperplane. Construct a correctly solvable problem for this case.

The aims and objectives of the study. Our main goal is to construct correct problems for the Laplace-Beltrami operator on the Riemannian sphere and to develop the spectral theory of some differential operator on Riemannian manifolds.

The main provisions for the defense of the dissertation:

- 1) Investigation of the well-posedness of Laplace-Beltrami operator on a punctured two-dimensional sphere and formulation of the maximal operator and its reversible restrictions. Investigation of Green's function of the Laplace-Beltrami operator with a finite number of punctured points. Investigation of spectral properties of the Laplace-Beltrami operator with delta-shaped perturbations;
- 2) Formulation of solvable problem for the Laplace-Beltrami operator on a two-dimensional sphere punctured by an open curve. Investigation of uniquely solvable problem for the Laplace-Beltrami operator on a two-dimensional sphere punctured by an open curve;
- 3) Formulation of solvable problem for the Laplace-Beltrami operator on a three-dimensional sphere punctured by a closed curve. Investigation of uniquely solvable problem for the Laplace-Beltrami operator on a three-dimensional sphere punctured by a closed curve;

4) Formulation the spectral problem for Laplace-Beltrami operator and perturbed harmonic oscillator. Investigation of the spectral properties of operators.

The objects of research are differential operators on Riemannian manifolds.

The research subjects are well-posedness and uniqueness of solvable problems, properties of their solutions.

Research methods. The research methods for addressing correctly solvable problems for the Laplace-Beltrami operator on a Riemannian sphere with cuts typically involve a combination of mathematical analysis, differential geometry, and numerical methods.

It is developed a mathematical framework for representing the Riemannian sphere with cuts. This involve using differential geometry to describe the surface, considering the existence of cuts as topological features, and defining the Laplace-Beltrami operator within this framework.

Also, the partial differential equation (PDE) involving the Laplace-Beltrami operator on the surface with cuts is formulated. This may include terms for the Laplace-Beltrami operator, boundary conditions, and any additional terms relevant to the problem under investigation.

The conditions for the existence and uniqueness of solutions to the PDE is proved. This step is crucial in establishing "correctly solvable problems." It often involves applying theorems from functional analysis and partial differential equations theory.

The topology of the Riemannian sphere with cuts is studied and the behavior of cuts, their locations, and their impact on the geometry of the surface is analyzed.

Novelty of the research. The research on "Correctly Solvable Problems for the Laplace-Beltrami Operator on a Riemannian Sphere with Cuts" introduces a novel and innovative approach to solving mathematical problems related to the Laplace-Beltrami operator on a Riemannian sphere with discontinuities or cuts.

The research addresses the Laplace-Beltrami operator in the context of Riemannian spheres that have discontinuities or cuts. This is a departure from traditional analyses that often assume smooth, continuous manifolds. The inclusion of cuts significantly broadens the scope of problems that can be tackled.

The term "correctly solvable problems" signifies a rigorous and well-defined approach to finding solutions. The research introduces a systematic framework for determining which problems involving the Laplace-Beltrami operator on a sphere with cuts can be correctly solved. This involves identifying the conditions under which solutions exist and are unique.

The study explores the interplay between the geometric and topological aspects of Riemannian spheres with cuts. This interaction is crucial in understanding the behavior of the Laplace-Beltrami operator in such non-standard settings, and it provides a deeper insight into the geometry of these surfaces.

While the research primarily focuses on theoretical developments, it has practical implications in various fields, such as physics, engineering, and computer science. Understanding the Laplace-Beltrami operator on surfaces with cuts can be relevant to problems involving heat diffusion, wave propagation, and data analysis on irregularly shaped surfaces.

The research introduces novel mathematical tools and techniques for solving partial differential equations on such surfaces. These tools can have broader applications in mathematics and may open up new research directions.

The presence of cuts introduces computational challenges when solving problems involving the Laplace-Beltrami operator. The research may address or propose new methods to overcome these challenges, which is an important contribution for researchers working on numerical simulations.

In summary, the novelty of this research lies in its exploration of Laplace-Beltrami operator problems on Riemannian spheres with cuts, its emphasis on correctly solvable problems, and its potential impact on mathematics and various applied fields. It offers a fresh perspective on differential equations in non-standard geometric settings and presents opportunities for further research and practical applications.

Theoretical and practical significance of the research. The importance of the dissertation research stems from the fact that the subjects of the study, namely the Laplace-Beltrami operator on a Riemannian manifold, have significant relevance not only within mathematical science but also in fields such as mechanics, physics, biology, and other natural sciences. Additionally, there is considerable interest in these problems from a mathematical perspective. Consequently, the findings are both relevant and comprehensible to researchers worldwide, and they can serve as a valuable foundation for future investigations.

Volume and structure of the dissertation. The thesis consists of the title page, contents, an introduction, preliminaries, four chapters, a conclusion and a list of references. Total volume of the dissertation is 100 pages with 61 literature references.

Main content of the dissertation. The introduction contains actuality of the research theme, aims and objectives, the main provisions for the defense of the dissertation, the research object and subject, connection of the dissertation thesis with the other scientific research works, the work approbation, author's publications, volume and structure of the dissertation and content.

In Chapter 1, delta-like perturbations of the Laplace-Beltrami operator on a Riemannian manifold without boundary are studied. Also, the two-dimensional sphere of \mathbb{R}^3 as a Riemannian manifold is considered. Similar constructions can be done for the Laplace-Beltrami operator on smooth manifolds without boundary. Other possibilities are connected with the description of correct restrictions of linear differential operators on smooth manifolds with boundary. Finally, the problem of describing well-posed problems for elliptic differential operators of derivative orders on manifolds with boundary consisting of components of different dimensions is of interest. Also, Green's function of the perturbed Laplace-Beltrami operator defined on a two-dimensional sphere is introduced. Descriptions of linear functionals for an arbitrary point from a punctured sphere are given. Expressions of Green's function $\varepsilon_\gamma(x, t)$ in terms of $\varepsilon(x, t)$ are given.

In Chapter 2, a fixed arc is removed from the sphere and investigation the additional conditions must satisfy the solution on the removed arc in order to guarantee the uniqueness of such a solution.

In Chapter 3, a circle on a three-dimensional sphere from four-dimensional space has been removed. It is required to describe well-solvable problems for the Laplace-Beltrami equation on the resulting domain. A class of such problems for the Laplace-Beltrami equation on a three-dimensional sphere with a cut along a circle is discussed in the work.

In Chapter 4, the eigenfunctions of the perturbed Laplace-Beltrami operator defined on a two-dimensional sphere is studied. Descriptions of linear functionals for an arbitrary point from a punctured sphere are given. The relationship between eigenvalues and operator eigenfunctions is shown, and a complete description of the system of eigenfunctions is given. In this work the discreteness property of the spectrum of the Laplace-Beltrami operator on a punctured two-dimensional sphere is proved. An example is also given for linear continuous functionals, where the first functional is defined through an integral, and the other two are equal to zero. The example is an explicit use of a theorem which proof used previously obtained results for a perturbed operator. The invariance of the Laplace-Beltrami operator with respect to isometries implies the coincidence of the spectrum of the Laplace-Beltrami operator of isometric Riemannian manifolds. Thus, the spectrum of the operator is an isometric invariant.

In the second part of Chapter 4, the eigenfunctions of a perturbed harmonic oscillator is studied. Descriptions of linear functionals for an arbitrary point are given. The relationship between eigenvalues and eigenfunctions of the operator, perturbed and unperturbed operators is shown. The property of discreteness of the spectrum of a harmonic oscillator on the straight axis is proved in the work. An example is also given where the first functional is defined through the integral, and the other five are equal to zero. The example is an explicit use of a theorem whose proof used previously obtained results for a perturbed operator.

In the conclusion, the main results obtained in the course of the dissertation thesis are presented.

Connection of the dissertation thesis with the other scientific research works. The dissertation thesis has been supported within the framework of a scientific project of the grant funding of young scientists under the "Zhas Galym" project for 2024-2026 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan «Well-posed problem for differential operators on a Riemannian manifold» (2024-2026 years, AP AP22685565). Corresponding to the directions of scientific development by priority: Intellectual potential of the country, by sub-priority: Fundamental and applied research in the field of mathematics, mechanics, astronomy, physics, chemistry, biology, computer science and geography.

Publications and author contributions. Based on the results of the dissertation 11 works were published: 6 journal articles, all of them in Scopus and Web of Science indexed Journals and 5 works in proceedings of international scientific conferences.

Some of them are the followings:

- 1) Kanguzhin B.E., Dosmagulova K.A. Well-posed problems for the Laplace–Beltrami operator on a punctured two-dimensional sphere // *Advances in the Theory of Nonlinear Analysis and Its Applications*. – 2023. – Vol. 7, No. 2. – P. 428–440.
- 2) Dosmagulova K., Kanguzhin B. Uniquely solvable problems for the Laplace–Beltrami operator on a sphere punctured by a curve // *Results in Nonlinear Analysis*. – 2023. – Vol. 6, No. 3. – P. 43–49.
- 3) Dosmagulova K.A., Kanguzhin B.E., Fazullin Z.Y. Spectrum of the perturbed harmonic oscillator // *Applied Mathematics and Information Sciences*. – 2023. – Vol. 17, No. 4. – P. 553–557.
- 4) Dosmagulova K.A., Kanguzhin B.E. Green’s function of the perturbed Laplace–Beltrami operator with a finite number of punctures on the two-dimensional sphere // In: *Women in Analysis and Partial Differential Equations. Series: Trends in Mathematics*. – Vol. 5. – Cham: Birkhäuser, 2024. – P. 145–151.
- 5) Kanguzhin B.E., Dosmagulova K.A., Akanbay Y.E. On the Laplace–Beltrami operator in stratified sets composed of punctured circles and segments // *Journal of Mathematics, Mechanics and Computer Science*. – 2025. – Vol. 125, No. 1. – P. 33–42.
- 6) Dosmagulova K.A., Kanguzhin B.E. Well-posed problems for the Laplace–Beltrami operator // *Symmetry*. – 2025. – Vol. 17, No. 9. – P. 1–12.